8.4 Multiple Angle Identities

Using the sum identities from the last section, we can derive more formulas called the double-angle identities:

$$\sin(2x) = \sin(x+x) = \sin x\cos x + \cos x\sin x = 2\sin x\cos x$$
$$\cos(2x) = \cos(x+x) = \cos x\cos x - \sin x\sin x = \cos^2 x - \sin^2 x$$
$$\cos(2x) = \cos^2 x - \sin^2 x = \left(1-\sin^2 x\right) - \sin^2 x = 1 - 2\sin^2 x$$
$$\cos(2x) = \cos^2 x - \sin^2 x = \cos^2 x - \left(1-\cos^2 x\right) = 2\cos^2 x - 1$$
$$\tan(2x) = \tan(x+x) = \frac{\tan x + \tan x}{1-\tan x\tan x} = \frac{2\tan x}{1-\tan^2 x}$$

Example: Find $\sin\frac{2\pi}{3}$, $\cos\frac{2\pi}{3}$, and $\tan\frac{2\pi}{3}$ using double-angle identities.

Example: Use the double angle identities to verify that $\cos(3x) = \cos^3 x - 3\cos x\sin^2 x$ is an identity.

We can use the double-angle identities to derive identities for $\sin(x/2)$, $\cos(x/2)$, and $\tan(x/2)$. We call these the half-angle identities.

To get identities for $\cos(x/2)$ and $\sin(x/2)$, we solve two of the equations for $\cos(2x)$ for $\sin x$ and $\cos x$.

$$2\cos^2 x - 1 = \cos(2x)$$
$$\cos^2 x = \frac{1+\cos(2x)}{2}$$
$$\cos x = \pm\sqrt{\frac{1+\cos(2x)}{2}}$$

$$1 - 2\sin^2 x = \cos(2x)$$
$$-2\sin^2 x = \cos(2x) - 1$$
$$\sin^2 x = \frac{1-\cos(2x)}{2}$$
$$\sin x = \pm\sqrt{\frac{1-\cos(2x)}{2}}$$

Since these equations work for any value of x, they also work if we replace x by $(x/2)$:

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \qquad \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

We can then use these formulas to derive formulas for $\tan\frac{x}{2}$:

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} \qquad \tan\frac{x}{2} = \frac{\sin x}{1+\cos x} \qquad \tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$$

Examples: Use the half-angle identities to find the exact values of $\sin\frac{\pi}{8}$, $\cos\frac{\pi}{8}$, and $\tan\frac{\pi}{8}$.

Example: Prove that $\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) = \frac{\sin^2 x}{4}$ is an identity.

Example: Find $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ if $\cos(2\alpha) = -1/3$ and $\pi < 2\alpha < 3\pi/2$.

Example: Find $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ if $\sin(\alpha/2) = 4/5$ and $\pi/4 < \alpha/2 < \pi/2$.

Solving multiple angle and half angle equations:

Often, equations involve expressions like $\sin 2x$, $\cos 3\alpha$, or $\tan(x/2)$, all of which involve multiples of the variable rather than a single variable. To solve these equations, we solve for the multiple variable just as we would solve for a single variable and then multiply or divide to get the single variable in the last step.

Example: Find all solutions in degrees to $\sin 2\alpha = \sqrt{3}/2$.

Example: Find all solutions to $\tan(4x) = 1$ in the interval $(0, \pi)$.

Example: Find all real number solutions to $\cos(x/2) = \sqrt{3}/2$.

Example: Find all solutions to $\csc(2x) = 2\sqrt{3}/3$ in the interval $(0°, 360°)$.